1.1.

Differential Equations

Definition:- An equation involving one independent variable and its derivatives with respect to one or more independent variable is called a differential equation.

Ordinary differential equation (ODE) contains only ordinary derivatives (not partial) and describes the relation between these dependent variable, say y, with respect to independent variable, say x.

Almost all the elementary and numerous advanced parts of theoritical physics, chemistry, biology, engineering economics and many other fields of applied science are formulated in terms of either ordinary differential eqns or partial differential equations. (PDE).

The solution of an, say. ODE may be written as

$$y = f(x) \equiv y(x)$$

For an ODE to have a closed form solution $y(x)$ can be expressed in terms of standard elementary functions like e^x, $\ln x$, $\sin x$ etc. For other ODE the solution $y(x)$ may not be in closed form but may be expressed as an infinite series.

If $y = f(x)$ one can form or calculate in general the derivative

$$y' = \frac{dy}{dx}$$

In a natural process the variables and their rate of changes often connected with one another by means of basic scientific principles that govern the process. The connection expressed in terms of mathematical symbols results often in a differential equation.

Example:

Newton's 2nd law:

$$ma = F$$

$$m\frac{d^2y}{dt^2} = F$$

For a body falling under gravity $m\frac{d^2y}{dt^2} = mg$ — (1·1)

or $\frac{d^2y}{dt^2} = g$

For an air resistance proportional to velocity we can write

$$m\frac{d^2y}{dt^2} = mg - k\frac{dy}{dx} \quad - (1:2)$$

These equations 1.1 and 1.2 are differential equations expressing the essential attributes of physical process under consideration.

Other examples: $\frac{dy}{dt} = -ky$

$$m\frac{d^2y}{dt^2} = -ky$$

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-p^2)y = 0$$

Order and degree of DE :-

The order of a differential equation is simply the order of the highest derivative it contains.

equations containing $\frac{dy}{dx}$ is of first order

$\frac{d^2y}{dx^2}$ " " 2nd order

$\frac{d^3y}{dx^3}$ " " 3rd "

and so on.

The degree of a differential equation is the power to which the highest derivative is raised [The equation should be rationalised to contain only integer powers of the derivative]

Ex 1 $\frac{d^3y}{dx^3} + x\left(\frac{dy}{dx}\right)^{3/2} + x^2y = 0 \Rightarrow$ Third order and 2nd degree.

Most general higher degree first order eqn

$F(x, y, \frac{dy}{dx}) = 0$

The most general standard form is

$p^n + a_{n-1}(x,y)p^{n-1} + \dots + a_1(x,y)p + a_0(x,y) = 0$

where $p = \frac{dy}{dx}$

We shall not discuss the solutions of such equations (if it can be solved).

Partial Differential Equations: A partial differential equation is one involving more than one independent variable so that the derivatives occuring are partial. A large class exists in physics

e.g. ① $\nabla^2\phi = 0$ Laplace's Eqn (Homogeneous)

occurs in the study of

(a) electromagnetic phenomena including electrostatics, dielectrics, magnetostatics & steady currents.

(b) hydrodynamics

(c) heat flow

(d) gravitation

② Poisson's equation (Inhomogeneous)

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

(3) The wave equation and time independent diffusion eq^n

$$\nabla^2\phi = \pm k^2\phi$$

occurs in the study of

a) elastic waves in solids, vibrating strings, bars and membranes.

b) sound or accoustics

c) electromagnetic waves

d) nuclear reaction

(4) Time dependent diffusion eqn

$$\nabla^2\phi = \frac{1}{a^2}\frac{\partial\phi}{\partial t}$$

(5) Time dependent wave eq^n

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

(6) schrödinger eq^n

$$-\frac{\hbar^2}{2m}\nabla^2\phi + V\phi = E\phi$$

or $$-\frac{\hbar^2}{2m}\nabla^2\phi + V\phi = i\hbar\frac{\partial\phi}{\partial t}$$

(7) The equations for viscous fluid and telegraphy eq^n.

(8) Maxwell's coupled partial differential equations for electric and magnetic field.

and a host of other complicated equation.

The most general ODE may be written as

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right) = 0$$

A similar type of functional form may be expressed for partial differential equations for which we have

$$H\phi = G$$

$$H = H\left(\frac{\partial \phi}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}, x, y, z\right)$$

G = known function.

Second order differential equations occur more frequently in physical applications than first order equations.

These equations are linear i.e. the operation satisfy

$$\hat{O}(\alpha\phi) = \alpha\hat{O}\phi$$

and $\hat{O}(\phi_1 + \phi_2) = \hat{O}\phi_1 + \hat{O}\phi_2$

The derivatives $\frac{d^n}{dx^n}$ and $\int [\;]\,dx$ are such examples.

Example of non linear operators

$(ax)^2$ and $\sin(\theta + \phi)$

$\because (ax)^2 \neq ax^2$ and $\sin(\theta+\phi) \neq \sin\theta + \sin\phi$

Examples of non linear operators:

a) Fundamental equation of atmospheric physics
b) Turbulence and chaos
c) shock waves.

General solution of an ODE is the most general fu^n $y(x)$ that satisfies the eq^n. It will contain constants of integration ~~—~~ that may be determined from boundary condition.

The general solutions of nth order ODE contains n (essential) arbitrary constants of integration. One needs n boundary conditions to determine the constants. When the boundary conditions are applied and the constants found, we get a particular solution to the ODE which obeys the given boundary conditions. Some of the ODE's of degree greater than one also possesses singular solutions which are solutions that contains no arbitrary constants. and can not be found from general solutions.

<u>Note</u>: When any solution to an ODE is found its validity can be checked by substitution into the original eq^n, checking that any boundary conditions are also met.